Chapter 10
Two or More Random Variables

10.1 **a.** Pr(X = 1 and Y = 3) = f_{XY} (1, 3) = 2/24 = 1/12

b. Pr(X + Y $\leq$ 4) = Pr(X = 0 and Y = 2) + Pr(X = 0 and Y = 3) + Pr(X = 0 and Y = 4)
+ Pr(X = 1 and Y = 2) + Pr(X = 1 and Y = 3) + Pr(X = 2 and Y = 2)
= 1/24 + 3/24 + 1/24 + 2/24 + 2/24 + 2/24 = 11/24

10.3 **a.** Pr(X = 1 | Y = 3) = Pr(X = 1 and Y = 3) / Pr(Y = 3) = (1/12)/(3/24 + 2/24 + 1/24) = 1/3

b. They are not independent because the joint probability table is not a multiplication table of the marginal probabilities. For example, f_{XY} (0, 0) = 1/24 but f_X (0) f_Y (1) = (6/24)(5/24) $\neq$ 1/24.

10.5 They are not independent because the joint probability table is not a multiplication table of the marginal probabilities. For example, f_{XY} (0, 1) = 0 but f_X (0) f_Y (1) = (1/32)(8/32) $\neq$ 0.
They are uncorrelated because Cov(X, Y) = E(XY) – E(X)E(Y) = 5 – (2.5)(2) = 0

10.9 E(X) = nπ, Var(X) = nπ(1– π)

E(Y) = E(n – X) = n – E(X) = n – nπ = n(1 – π)

Var(Y) = nπ(1 – π)

Var(X + Y) = Var(X) + 2Cov(X, Y) + Var(Y) = Var(n) = 0

So Cov(X, Y) = –(1/2)[Var(X) + Var(Y)] = –nπ(1 – π)

Corr(X, Y) = Cov(X, Y)/$\sigma_X \sigma_Y$ = –nπ(1 – π)/[nπ(1 – π)] = –1

10.11 **a.** According to Table 10.1:

$f_{XY}(x, y)$		y 0	1	2	3	4	$f_X(x)$
	0	1/32	0	0	0	0	1/32
	1	0	2/32	3/32	0	0	5/32
x	2	0	2/32	3/32	4/32	1/32	10/32
	3	0	2/32	3/32	4/32	1/32	10/32
	4	0	2/32	3/32	0	0	5/32
	5	1/32	0	0	0	0	1/32
$f_Y(y)$		2/32	8/32	12/32	8/32	2/32	1

E(X) = 0 + (1)(5/32) + (2)(10/32) + (3)(10/32) + (4)(5/32) + (5)(1/32) = 5/2

E(Y) = 0 + (1)(8/32) + (2)(12/32) + (3)(8/32) + (4)(2/32) = 2

$E(X^2) = 0 + (1^2)(5/32) + (2^2)(10/32) + (3^2)(10/32) + (4^2)(5/32) + (5^2)(1/32) = 15/2$

$E(Y^2) = 0 + (1^2)(8/32) + (2^2)(12/32) + (3^2)(8/32) + (4^2)(2/32) = 5$

E(XY) = 0 + (1)(2/32) + (2)(3/32) + (2)(2/32) + (4)(3/32) + (6)(4/32) + (8)(1/32) + (3)(2/32) + (6)(3/32) + (9)(4/32) + (12)(1/32) + (4)(2/32) + (8)(3/32) = 5

$Var(X) = E(X^2) – [E(X)]^2 = 15/2 – (5/2)^2 = 5/4$

$Var(Y) = E(Y^2) – [E(Y)]^2 = 5 – 2^2 = 1$

Cov(X, Y) = E(XY) – E(X)E(Y) = 5 – (5/2)(2) = 0

So, E(X + Y) = E(X) + E(Y) = 5/2 + 2 = 9/2 = 4.5

Var(X + Y) = Var(X) + Var(Y) + 2 Cov(X, Y) = 5/4 + 1 – 2(0) = 2.25

b. ρ(X, Y) = Corr(X, Y) = 0

10.15 $E(X_j) = 6/10$ for j = 1 to10.
So, $E(K) = E(X_1) + E(X_2) + \ldots + E(X_{10}) = (10)(6/10) = 6$

a. $E(X_1X_2) = 5/15$.
So $Cov(X_1, X_2) = E(X_1X_2) – E(X_1)E(X_2) = 5/15 – (6/10)^2 = –2/75$

b. $Var(X_1) = Var(X_2) = (6/10)(4/10) = 6/15$
$Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2\ Cov(X_1, X_2) = (2)(6/25) + (2)(–2/75) = 32/75$

10.17 The joint probability table with marginal probabilities is:

		x			
$f_{XY}(x, y)$		10	15	20	$f_Y(y)$
y	5	0.20	0.10	0.05	0.35
	10	0.05	0.20	0.30	0.55
	15	0.02	0.04	0.04	0.10
$f_X(x)$		0.27	0.34	0.39	1

a. E(X) = (10)(.27) + (15)(.34) + (20)(.39) = 15.60
E(Y) = (5)(.35) + (10)(.55) + (15)(.10) = 8.75
E(X + Y) = E(X) + E(Y) = 15.60 + 8.75 = 24.35

b. E(2X – 3Y) = 2E(X) – 3E(Y) = 2(15.60) – 3(8.75) = 4.95

c. Pr(18 < X + Y < 27) = Pr(X = 10 and Y = 10) + Pr(X = 10 and Y = 15) + Pr(X = 15 and Y = 5) + Pr(X = 15 and Y = 10) + Pr(X = 20 and Y = 5) = .05 + .02 + .10 + .20 + .05 = 0.42.

10.19

		x			
$f_{XY}(x, y)$		10	15	20	$f_Y(y)$
y	5	0.20	0.10	0.05	0.35
	10	0.05	0.20	0.30	0.55
	15	0.02	0.04	0.04	0.10
$f_X(x)$		0.27	0.34	0.39	1

a. $E(X^2) = (10^2)(.27) + (15^2)(.34) + (20^2)(.39) = 259.50$

b. $E(Y^2) = (5^2)(.35) + (10^2)(.55) + (15^2)(.10) = 86.25$

c. E(XY) = (50)(.20) + (75)(.10) + (100)(.05) + (100)(.05) + (150)(.20) + (200)(.30) + (150)(.02) + (225)(.04) + (300)(.04) = 141.5

d. E(X) = (10)(.27) + (15)(.34) + (20)(.39) = 15.60

E(Y) = (5)(.35) + (10)(.55) + (15)(.10) = 8.75

Cov(X, Y) = E(XY) – E(X)E(Y) = 141.5 – (15.60)(8.75) = 5

e. Var(X) = E(X^2) – (E(X))^2 = 259.50 – 15.60^2 = 16.14

Var(Y) = E(Y^2) – (E(Y))^2 = 86.25 – 8.75^2 = 9.6875

Therefore, Corr(X, Y) = Cov(X, Y)/σ_X σ_Y = 5/[(16.14)(9.6875)]$^{1/2}$ = 0.3999

f. X and Y are not independent because the joint probability table is not a multiplication table of the marginal probabilities.

For example, f_{XY} (10, 15) = 0.20 but f_X (10) f_Y (15) = (0.27)(0.35) $\neq$ 0.20

10.21 The joint probability table with marginal probabilities is:

	f_{XY} (x, y)	x = 0	1	2	3	f_Z (z)
	0	0	0	0.15	0.05	0.20
z	1	0	0.30	0.30	0	0.60
	2	0.05	0.15	0	0	0.20
	f_X (x)	0.05	0.45	0.45	0.05	1

a. E(X) = 0 + (1)(.45) + (2)(.45) + (3)(.05) = 1.5

E(Z) = 0 + (1)(.60) + (2)(.20) = 1

E(X + Z) = E(X) + E(Z) = 1.5 + 1 = 2.5

b. E(X · Z) = (1)(.3) + (2)(.30) + (2)(.15) = 1.2

c. Cov(X, Z) = E(X · Z) – E(X) E(Z) = 1.2 – (1.5)(1) = – 0.3

d. E(X^2) = 0 + (1^2)(.45) + (2^2)(.45) + (3^2)(.05) = 2.7

E(Z^2) = 0 + (1^2)(.60) + (2^2)(.20) = 1.4

Var(X) = E(X^2) – (E(X))^2 = 2.7 – $(1.5)^2$ = 0.45

Var(Z) = E(Z^2) – (E(Z))^2 = 1.4 – $(1)^2$ = 0.40

Var(X + Z) = Var(X) + 2Cov(X, Z) + Var(Z) = .45 + 2(–0.3) + 0.40 = 0.25

10.23 Let X = verbal score, Y = quantitative score, then X ~ Normal(50, 20), Y ~ Normal(100, 10).
E(X) = 50, σ_X = 20, E(Y) = 100, σ_Y = 10, Cov(X, Y) = 75

a. E(C) = E(3X + 2Y) = 3E(X) + 2E(Y) = 3(50) + 2(100) = 350

$Var(C) = 3^2 Var(X) + 2(3)(2)Cov(X, Y) + 2^2 Var(Y) = 9(20^2) + 12(75) + 4(10^2) = 4900$

Therefore, $\sigma_C = 70$

b. $Pr(C \geq 375) = Pr(Z \geq [(375 - 350)/70] = Pr(Z \geq 0.3571) = 0.3605$
Therefore, about 36.05% of the applicants get a composite score of at least 375.

Remark: First printing of book had standard deviations interchanged, which resulted in $\sigma_C = \sqrt{3400}$ and $Pr(C \geq 375) = .3341$.

10.25 Y_1, Y_2, Y_3 is a random sample of three elements with probability function *f(y)*.
The random variables Y_1, Y_2, and Y_3 are independent.
So the joint probability function of Y_1, Y_2, and Y_3 is $f(y_1, y_2, y_3) = f(y_1) f(y_2) f(y_3)$.

10.27 **a.** $E(Y) = aE(Z_1) + bE(Z_2) = a(0) + b(0) = 0$

b. $Var(Z_1) = E(Z_1^2) - (E(Z_1))^2 = 1$, so $E(Z_1^2) = 1$
Because Z_1 and Z_2 are independent, $Cov(Z_1, Z_2) = 0$, $E(Z_1Z_2) = E(Z1)E(Z2) = 0$

$Var(Y) = a^2 Var(Z_1) + b^2 Var Z_2) = a^2(1) + b^2(1) = a^2 + b^2$

c. $Cov(X, Y) = E(XY) - E(X)E(Y) = E(Z_1(aZ_1 + bZ_2)) + E(Z_1)E(Y)$
$= E(a Z_1^2 + b Z_1Z_2) + 0 = aE(Z_1^2) + bE(Z_1Z_2) = a(1) + b(0) = a$
So $Corr(X, Y) = Cov(X, Y)/ \sigma_X\sigma_Y = a / \sqrt{(1)(a^2 + b^2)} = a / \sqrt{a^2 + b^2}$
(assuming a and b are not both zero)

10.29 $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$

a. When $0 \le x \le 1$ and $0 \le y \le 1$, $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y}(xy) = 1$

When $0 \le x \le 1$ and $y > 1$, $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y}(x) = 0$

When $x > 1$ and $0 \le y \le 1$, $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y}(y) = 0$

When $x > 1$ and $y > 1$, $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y}(1) = 0$

When $x < 0$ and $y < 0$, $f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) = \frac{\partial^2}{\partial x \partial y}(0) = 0$

Therefore, $f_{XY}(x, y) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

b. $f_X(x) = \int_{-\infty}^{+\infty} fxy(x, y)dy = \begin{cases} \int_{-\infty}^{0} 0dy + \int_{0}^{1} 1dy + \int_{1}^{\infty} 0dy = 1 & \text{if } 0 \le x \le 1 \\ \int_{-\infty}^{+\infty} 0dy = 0 & \text{otherwise} \end{cases}$

c. $f_Y(y) = \int_{-\infty}^{+\infty} fxy(x, y)dx = \begin{cases} \int_{-\infty}^{0} 0dx + \int_{0}^{1} 1dx + \int_{1}^{\infty} 0dx = 1 & \text{if } 0 \le y \le 1 \\ \int_{-\infty}^{+\infty} 0dx = 0 & \text{otherwise} \end{cases}$

So $f_{XY}(x, y) = f_X(x) f_Y(y)$. Therefore, A and Y are independent.

d. $E(X) = \int_{-\infty}^{+\infty} xfx(x)dx = \int_{-\infty}^{0} x(0)dx + \int_{0}^{1} x(1)dx + \int_{1}^{\infty} x(0)dx = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$

$E(X^2) = \int_{-\infty}^{+\infty} x^2 fx(x)dx = \int_{-\infty}^{0} x^2(0)dx + \int_{0}^{1} x^2(1)dx + \int_{1}^{\infty} x^2(0)dx = \frac{1}{3}(1^3 - 0^3) = \frac{1}{3}$

$Var(X) = E(X^2) - E^2(X) = (1/3) - (1/2)^2 = 1/12$

Therefore, $\sigma_X = (Var(X))^{1/2} = (1/12)^{½} = 0.2887$

e. $Pr(0 < X \le 1/2 \text{ and } 1/2 < Y < 1) = Pr(0 < X \le 1/2)\, Pr(1/2 < Y < 1)$

$= \int_{0}^{1/2} fx(x)dx \int_{1/2}^{1} fy(y)dy = \int_{0}^{1/2} (1)dx \int_{1/2}^{1} (1)dy = (1/2 - 0)(1 - 1/2) = 1/4 = 0.25$