Chapter 3
Probability

3.1 The program depends on the technology you use. Here is one that works on the TI-83 Plus graphing calculator. It takes a minute or so to run on the TI-83.

```
PRGM→NEW
Name=LAB1
0 STO→ C
0 STO→ I
FOR (I,1,100,1)
RAND STO→  X
RAND STO→  Y
IF (X^2 + Y^2) < 1
C+1 STO→ C
END
DISP  4*C/100
QUIT
```

Notes: Variable **C** is for the number of successes;
I is for the number of trials;
FOR is under **PRGM → CTL**;
RAND is under **MATH → PRB**;
IF is under **PRGM → CTL**;
< is under **TEST → TEST**;
END is under **PRGM → CTL**;
DISP is under **PRGM → I/O**.

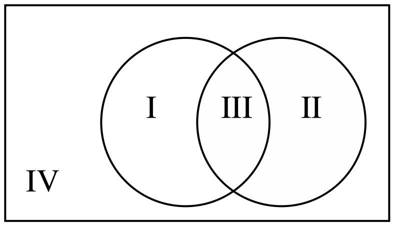
3.3 **a.** Pr(Three of four are heads) = Pr(HHHT or HHTH or HTHH or THHH)
= Pr(HHHT) + Pr(HHTH) + Pr(HTHH) + Pr(THHH)
$= (1/2)^3 (1/2)^1 + (1/2)^3 (1/2)^1 + (1/2)^3 (1/2)^1 + (1/2)^3 (1/2)^1 = 1/4$

b. Pr(Last two are heads) = (1/2) (1/2) = 1/4

c. Pr(The third is heads) = 1/2

d. Pr(All are the same) = Pr(All are heads) + Pr(All are tails) = $(2)\ (1/2)^4 = 1/8$

3.5



Let E = I ∪ III, F = II ∪ III, E ∪ F = I ∪ II ∪ III.
Then E ∩ F = III, Pr(E) = 0.35 + 0.55 – 0.75 = 0.15
E ∩ F′ = I, Pr(E ∩ F′) = 0.35 – 0.15 = 0.2
F ∩ E′ = II, Pr(F ∩ E′) = 0.55 – 0.15 = 0.4
(E ∪ F)′ = IV, Pr((E ∪ F)′) = 1 – 0.75 = 0.25.

3.7 Pr(female) = 99589/191596 = 0.5198
Pr(married with spouse absent) = 6822/191596 = 0.0356
Pr(female | married with spouse absent) = 4026/6822 = 0.5901
Pr(female and married with spouse absent) = 402/191596 = 0.0210

The event "female and married with spouse absent" considers the people who are both female and married with spouse absent within the entire population. Note that the outcome set for this event is the entire U.S. population, and about 2% of them are female and married with spouse absent.

The event "female | married with spouse absent" considers only the females within the subpopulation of people who are married with spouse absent. Here the outcome set consists of 6822 thousand, which is much smaller than the entire U.S. population of 191,596 thousand; and about 59% of them are female.

3.9 **a.** Pr(E and F) = Pr(E)Pr(F | E) = (.50)(.30) = .15

b. Pr(E or F) = Pr(E) + Pr(F) – Pr(F and E) = .50 + .20 – .15 = .55

c. Pr(Not E and Not F) = 1 – Pr(E or F) = 1 – .55 = .45

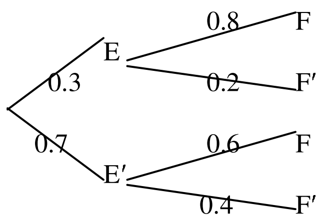
d. Pr(E | F) = Pr(E and F)/Pr(F) = .15/.20 = .75

3.11 Pr(At least one ace) = 1 – Pr(No ace) = $1 - (5/6)^4$
Addition rule cannot be used because events are not disjoint.

3.13 Pr(Snake eyes in a roll) = $(1/6)^2$ = 1/36,
Pr(No snake eyes in a roll) = 1 – (1/36) = 35/36,
Pr(No snake eyes in 24 rolls) = $(35/36)^{24}$, therefore
Pr(Snake eyes at least once in 24 rolls) = 1 – Pr(No snake eyes in 24 rolls) = $1 - (35/36)^{24}$.

3.15 By multiplication rule: Pr(E and F) = Pr(E) Pr(F | E)
Also Pr(E and F) = Pr(F) Pr(E | F) = Pr(E) Pr(F)
Therefore Pr(F | E) = Pr(F).

3.17 Make a tree diagram with first branching for E and E′ and the second branching for F and F′



EVENT	PROBABILITY
E and F	(0.3)(0.8) = 0.24
E and F′	(0.3)(0.2) = 0.06
E′ and F	(0.7)(0.6) = 0.42
E′ and F′	(0.7)(0.4) = 0.28

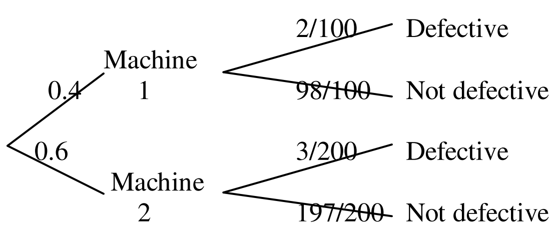
3.19 Pr(At least two share a common birthday) = 1 – Pr(Nobody shares a common birthday)
= 1 – (364/365)(363/365) ……(356/365) = 1 – (364!)/{(355!)(365^9)} = 1 – 0.88305 = 0.11695

3.23 **a. PV+ =** Pr(Cancer | Positive) = Pr(Cancer and Positive)/Pr(Positive)
= (1/63)(.8)/{(1/63)(.8) + (62/63)(.1)} = 8/(8 + 62) = 4/35

b. Pr(Cancer before test) = 1/63 = 0.0159
Pr(Cancer after a positive mammogram) = PV^+ = 4/35 = 0.1143

c. Pr(Cancer before test) = 1/63 = 0.0159
Pr(Cancer | Negative) = Pr(Cancer and Negative)/Pr(Negative)
= (1/63)(.2)/{(1/63)(.2) + (62/63)(.9)} = 0.0036

3.25

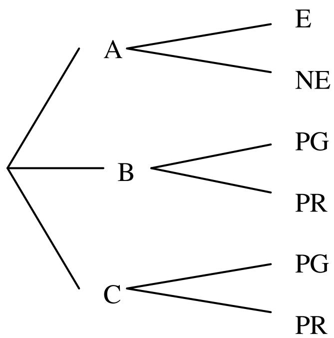


(.4)(2/100) = .008
(.4)(98/100) = .392
(.6)(3/200) = .009
(.6)(197/200) = .591

a. Pr(Made by machine 1 | Defective) = .008/(.008 + .009) = 0.4706

b. Pr(Made by machine 2 | Defective) = .009/(.008 + .009) = 0.5294

3.27 Make a tree diagram with first branching for the three different sources A, B, and C and the second branching E (messages exceeding 500 bytes in length) and NE (messages not exceeding 500 bytes in length).



EVENT	PROBABILITY
A & E	(.2)(.80) = .16
A & NE	(.2)(.2) = .04
B & E	(.3)(.6) = .18
B & NE	(.3)(.4) = .12
C & E	(.5)(.4) = .2
C & NE	(.5)(.6) = .3

a. Pr(Exceed 500 bytes) = .16 + .18 + .2 = 0.54

b. Pr(From source A | Exceed 500 bytes) = .16 / .54 = 0.2963
c. Pr(From source B | Exceed 500 bytes) = .18 / .54 = 0.3333

d. Pr(From source C | Exceed 500 bytes) = .2 / .54 = 0.3704

3.29 **a.** Pr(Test negative | Infected) = Pr(Infected) – Pr(Test positive | Infected) = x – (x)Sensitivity = x (1 – Sensitivity),
Pr(Test positive | Not infected) = Pr(Not infected) – Pr(Test negative | Infected) = (1 – x) – (1 – x)Specificity = (1 – x)(1 – Specificity)

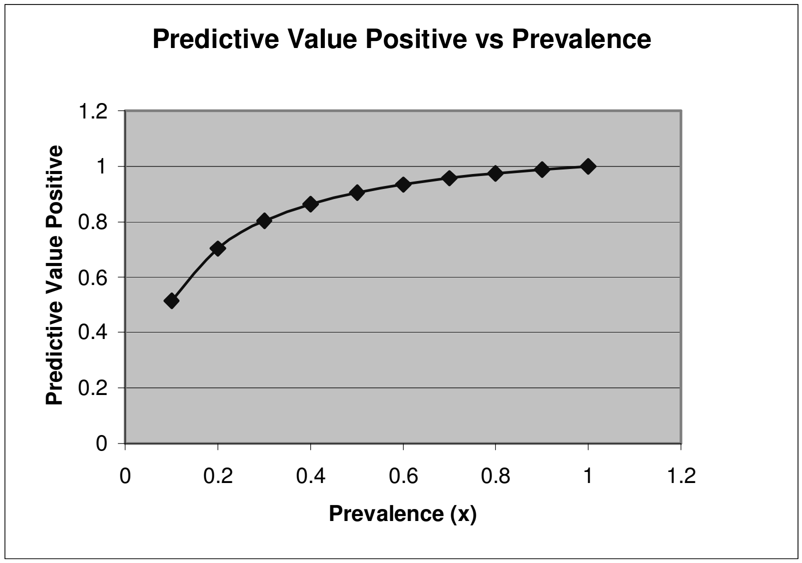
b. Pr(Test positive) = Pr(Test positive | Infected) + Pr(Test positive | Not infected) = (x)Sensitivity + (1 – x)(1 – Specificity)

c. PV^+ = Pr(Test positive | Infected) / Pr(Test positive) = (x)Sensitivity/{(x)Sensitivity + (1 – x)(1 – Specificity)

d. When Specificity = 0.9, Sensitivity = 0.95, $PV^+ = (.95x)/(.85x + .1)$
The predictive values positive with different values of x are:

x	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
PV^+	.5135	.7037	.8028	.8636	.9048	.9344	.9568	.9744	.9884	1

The graph of the predictive value positive as a function of x is shown below. There is an obvious pattern in the curve that the value of PV^+ converges to 1 as the value of x increases. Furthermore, for any $x \in (0, 1)$, the PV^+ always exceed the prevalence of the disease.



3.33 Statements **a** and **b** are always true, but statement **c** is not always true.

3.35 **a.** Pr(E and F) = Pr(E) Pr(F) = (0.20)(0.50) = 0.10

b. Pr(E or F) = Pr(E) + Pr(F) – Pr(E and F) = 0.20 + 0.50 – 0.10 = 0.60

c. Pr(F and Not E) = Pr(F) Pr(Not E) = (0.50)(1 – 0.20) = 0.40

d. Pr(Not E and Not F) = 1 – Pr(E or F) = 1 – 0.60 = 0.40

3.37 **a.** Pr(HTHH) = $(1/2)^4$ = 1/16 = 0.062

b. Pr(No heads) = Pr(Four tails) = Pr(TTTT) = $(1/2)^4$ = 1/16 =0.0625

c. Pr(At least one head) = 1 – Pr(No heads) = 1 – 1/16 = 15/16 = 0.9375

d. Pr(Four heads) = Pr(HHHH) = 1/16 = 0.0625

3.39 **a.** Pr(Both balls are red) = Pr(First is red)Pr(Second is red) = (2/6)(2/6) = 1/9

b. Pr(Both balls are white) = Pr(First is white)Pr(Second is white) = (4/6)(4/6) = 4/9

c. Pr(Both same color) = Pr(Both are red) + Pr(Both are white) = 1/9 + 4/9 = 5/9

d. Pr(They are of different color) = 1 – Pr(Both same color) = 1 – 5/9 = 4/9

e. If both balls are drawn without replacement:
Pr(Both balls are red) = Pr(First is red)Pr(Second is red) = (2/6)(1/5) = 1/15
Pr(Both balls are white) = Pr(First is white)Pr(Second is white) = (4/6)(3/5) = 2/5
Pr(Both same color) = Pr(Both are red) + Pr(Both are white) = 1/15 + 2/5 = 7/15
Pr(They are of different color) = 1 – Pr(Both same color) = 1 – 7/15 = 8/15

3.41 **a.** Pr(Both letters are s) = Pr(First is s)Pr(Second is s) = (2/11)(1/10) = 1/55

b. Pr(Both letters are the same) =
Pr(Both letters are s) + Pr(Both letters are t) + Pr(Both letters are a) + Pr(Both letters are i) =
(2/11)(1/10) + (3/11)(2/10) + (2/11)(1/10) + (2/11)(1/10) = 1/55 + 3/5 + 1/55 + 1/55 = 6/55

c. Pr(Both letters are s | Both letters are the same) =
Pr(Both letters are the same and Both letters are s)/Pr(Both letters are the same) =
(1/55)(6/55) = 1/6

3.43 Pr(Different numbers) = (5/6)(4/6) = 5/9

3.45 **a.** If two events are <u>disjoint</u> and you want to find the probability that <u>either one</u> will happen, you can <u>add</u> the probabilities.

b. If two events are <u>independent</u> and you want to find the probability that <u>both</u> will happen, you can <u>multiply</u> the probabilities.

3.47 Pr(At least one break down) = 1 – Pr(None break down) = 1 – $(.8)^{10}$ = 1 – .1073 = 0.8926

3.49 Pr(The smallest is 6) = $\dfrac{\binom{4}{2}\binom{1}{1}\binom{5}{0}}{\binom{10}{3}} = 1/20$
(choose 2 of the 4 badges 7,8,9,10; choose the badge 6; choose 0 of the 5 badges 1,2,3,4,5).

3.51 **a.** In order for the Cubs to win in a 6-game series, the Cubs must win the 6^{th} game, and another 3 of the first 5 games, and the White Sox win two of the first 5 games. Find the number of 6-letter words can be made with the first five letters consisting of 3 C's and 2 S's, and the sixth letter a C. So the number of ways that the Cubs win in the 6-game series is equal to the number of ways the Cubs win 3 of the first 5 games in the series, namely CCCSSC, CCSCSC, CSCCSC, SCCCSC, CCSSCC, CSCSCC, SCCSCC,CSSCCC,SCSCCC,SSCCCC: $\binom{5}{3} = \dfrac{5!}{3!2!} = 10$

b. With the same reasoning as part a, the number of ways that the Cubs win in a 5-game series is equal to the number of ways the Cubs win 3 and lose 1 of the first 4 games in the series, namely:

CCCSC, CCSCC, CSCCC, SCCCC: $\binom{4}{3} = \frac{4!}{3!1!} = 4$

c. Because the teams are evenly matched, the probability the Sox win the first game is 0.5.

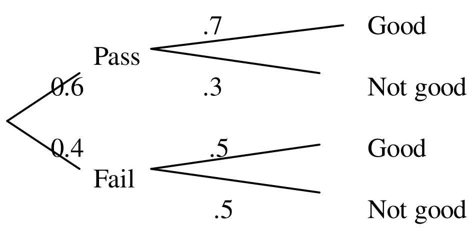
d. There are 1 + 4 + 10 + 20 = 35 ways that the Sox can win the series (1 way in a 4-game series, 4 ways in a 5-game series, 10 ways in a 6-game series, and 20 ways in a 7-game series). Similarly, there are 35 ways that the Cubs can win the series. The probability the Sox win the series is 35/70 = 0.5.

e. Suppose that the Sox have already won the first game. The Cubs can still win the series in a 5-game series, a 6-game series, or a 7-game series. The only outcome in which the Cubs win in a 5-game series is SCCCC, with (conditional) probability $(1/2)^4 = 1/16$. There are four outcomes in which the Cubs win in a 6-game series, of the form SXXXXC, where three of the X's are C's and the other X is an S; this (conditional) probability is $4(1/2)^5 = 1/8$. Similarly, there are ten outcomes of the form SXXXXXC, where three of the X's are C's and the other two are S's; this (conditional) probability is $10(1/2)^6 = 5/32$. Thus, if the Sox have already won the first game, the probability that the Cubs win the series is 1/16 + 1/8 + 5/32 = 11/32 = 0.34375.

3.53 **a.** Make a tree and use Bayes' method. Pr(HHH) = Pr(HHH and Fair) + Pr(HHH and Bad coin) = (4/5)(1/8) + (1/5)(1) = 0.30.
Pr(Fair | HHH) = Pr(HHH and Fair)/Pr(HHH) = 0.10/0.30 = 1/3

b. Make a tree and use Bayes' method. Pr(Six heads) =
Pr(Six heads and Fair) + Pr(Six heads and Bad coin) = (4/5)(1/64) + (1/5)(1) = 17/80.
Pr(Fair | Six heads) = Pr(Six heads and Fair)/Pr(Six heads) = (1/80)/(17/80) = 1/17

3.55 The tree diagram of the problem could be as follows:

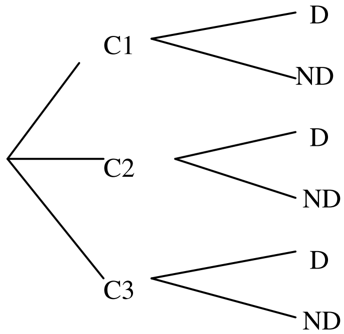


EVENT	PROBABILITY
Pass & Good	(.6)(.7) = .42
Pass & Not good	(.6)(.3) = .18
Fail & Good	(.4)(.5) = .20
Fail & not good	(.4)(.5) = .20

a. Percentage of the teachers rated as good is .42 + .20 = 0.62

b. Pr(Fail | Good) = Pr(Fail and Good)/Pr(Good) = .2/.62 = 0.3226

3.57 The tree diagram could be (D = defective, ND = not defective):



EVENT	PROBABILITY
C1 & D	(.1)(.01) = .001
C1 & ND	(.1)(.99) = .099
C2 & D	(.5)(.03) = .015
C2 & ND	(.5)(.97) = .485
C3 & D	(.4)(.04) = .016
C3 & ND	(.4)(.96) = .384

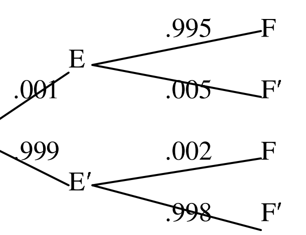
Pr(Defective) = .001 + .015 + .016 = 0.032

a. Pr(From C1 | Defective) = Pr(From C1 and Defective)/Pr(Defective) = .001/.032 = 0.0313

b. Pr(From C2 | Defective) = Pr(From C2 and Defective)/Pr(Defective) = .015/.032 = 0.4688

c. Pr(From C3 | Defective) = Pr(From C3 and Defective)/Pr(Defective) = .016/.032 = 0.5000

3.59 Make a tree diagram with first branching for E (With the disease) and E′ (Free of the disease) and the second branching for F (Test positive) and F′ (Test negative):

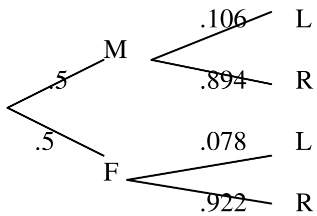


EVENT	PROBABILITY
E and F	$(.001)(.995) = 9.950 \cdot 10^{-4}$
E and F′	$(.001)(.005) = 5.000 \cdot 10^{-6}$
E′ and F	$(.999)(.002) = 1.998 \cdot 10^{-3}$
E′and F′	(.999)(.998) = 0.997

PV+ = Pr(Disease | Positive) = Pr(Disease and Positive)/Pr(Positive)
= {(.001)(.995)}/{(.001)(.995) + (.999)(.002)} = 995/2993 = 0.3324

PV− = Pr(No disease | Negative) = Pr(No disease and Negative)/Pr(Negative)
= {(.999)(.998)}/{(.999)(.998) + (.001)(.005)} = 997002/997007 = 0.999995

3.61 Make a tree diagram with first branching for M (male) and F (female) and the second branching for L (left-handed) and R (right-handed):



EVENT	PROBABILITY
M and L	(.5000)(.1060) = 0.0530
M and R	(.5000)(.8940) = 0.4470
F and L	(.5000)(.0780) = 0.0390
F and R	(.5000)(.9220) = 0.4610

Then Pr(L) = .0530 + .0390 = 0.0920
Therefore Pr(M | L) = Pr(M and L)/Pr(L) = 0.0530/0.0920 = 0.5761

3.63 Since the correct key could be any one of the seven keys with equal probability, the probability that the correct key is selected on the second draw is 1/7.

3.65 For the first selected committee member, the probability that he/she is a Democrat is 50/(50 + 49 +1) = 50/100.

For the second selected committee member, the probability that he/she is a Democrat is 49/(49 + 49 +1) = 49/99.

Therefore, Pr(Five members are Democrats) = (50/100)(49/99)(48/98)(47/97)(46/96) = 0.02814

3.67 Pr(The device fails to work) = Pr(At least one component functions incorrectly) =
1 – Pr(No components function incorrectly) = 1 – Pr(All components function correctly) =
$1-(.99)^5 = 0.0490$

3.69 Pr(A) = Pr(A infested) = 0.20, Pr(B) = Pr(B infested) = 0.50, $Pr(A \cup B)$ =
Pr(At least one infested) = 0.60. Then $Pr(A \cap B)$ = (Both infested) = 0.20+ 0.50 – .60 = 0.10

Therefore, Pr(A infested | B infested) = Pr(A | B) = $Pr(A \cap B)/Pr(B)$ = 0.10/0.50 = 0.2

3.71 The total number of combinations of the 10 computers is $2^{10} = 1024$.
In order for the system to remain up, four or more computers should be up. Therefore, the number of computer combinations in which the system will be up could be calculated as:

$$\binom{10}{4}+\binom{10}{5}+\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10} = 210 + 252 + 210 + 120 + 45 + 10 +1 = 848$$

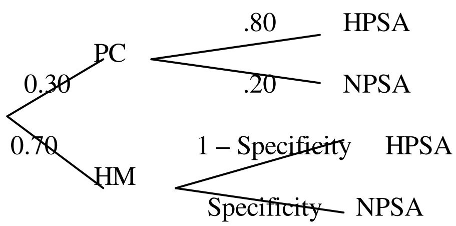
Or we can compute the number of ways the system can fail, and subtract it from 1024:

$$1024-\left\{\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}\right\} = 1024 - \{1 + 10 + 45 + 120\} = 848$$

3.73 **a.** Pr(Rated guilty) = Pr(PG and Rated guilty) + Pr(PI and Rated guilty)
= (25/81)(20/25) + (56/81)(6/56) = 26/81 = 0.321

b. Pr(PG | Rated guilty) = Pr(PG and Rated guilty/Pr(Rated guilty)
= {(25/81)(20/25)}/{(26/81)} = 10/13 = 0.769

3.75 Make a tree diagram with first branching for PC (men with prostate cancer) and HM (healthy men) and the second branching for HPSA (high levels of PSA) and NPSA (normal PSA levels):



EVENT	PROBABILITY
PC & HPSA	(.30)(.80) = .24
PC & NPSA	(.30)(.20) = .06
HM & HPSA	(.70)(1 – Specificity)
HM & NPSA	(.70)(Specificity)

a. Prevalence = 0.30

b. Sensitivity = 0.80

c. Pr(HPSA) = Pr(PC and HPSA) + Pr(HM and HPSA) = .24 + (.70)(1 – Specificity),
Pr(HM|HPSA) = Pr(HM and HPSA) / Pr(HPSA) =
{(.7)(1 – Specificity)}/{.24 + (.70)(1 – Specificity)} = 2/3.
Therefore, Specificity = 22/70

d. PV^+ = Pr(PC|HPSA) = 1 – Pr(HM|HPSA) = 1 – 2/3 = 1/3

e. PV^- = Pr(HM|NPSA) = Pr(HM & NPSA)/Pr(NPSA) =
(.7)(22/70)/{.06 + (.7)(22/70)} = 11/14 = 0.7857